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SOBRE LA NATURALEZA DE LA MATEMÁTICA Y SU PAPEL EN LA CIENCIA Y LA SOCIEDAD

ON THE NATURE OF MATHEMATICS AND ITS ROLE IN SCIENCE AND OUR SOCIETY

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Resumen

La lógica y la matemática son los productos naturales del desarrollo de la inteligencia. Si entendemos la matemática como el simbolismo de las acciones del sujeto sobre el objeto, entonces, tanto la verdad en la matemática como en la física tienen el mismo origen: se basan en nuestras interacciones con la realidad. Esto explica por qué no importa cuán abstracta es la matemática; siempre habrá una parte que se aplicará a la descripción física del mundo. La verdadera utilidad de la matemática reside en cambiar la mente revelando nuevos aspectos de la realidad y extender nuestra conciencia.

Abstract

Logic and mathematics are the natural consequences of the development of intelligence. If we conceive mathematics as the symbolism of the actions of the subject on the object, then the truths in physics and mathematics have exactly the same origin: they emanate from our interactions with reality. This explains why however abstract mathematics becomes, there will always be some branches of it that applies to the physical world. The real utility of mathematics is in changing our mind, by revealing new aspects of reality and extending our consciousness.

Palabras clave: matemáticas, ciencia, epistemología.

Keywords: mathematics, science, epistemology.

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Asking the right questions

What is mathematics and what is its usefulness for our society? During the middle ages the English natural philosopher Roger Bacon (ca. 1219-1292), who was also a Franciscan friar, considered the study of mathematics as essential, stating that "(…) he who is ignorant of it cannot know the other sciences or the things of this world" (Merzbach & Boyer, 2011, p. 223). On the same theme during the 17th century the Cambridge mathematician Isaac Barrow (1630-1677), whose work inspired Isaac Newton (1643-1727), described mathematics as, "(…) the unshaken foundation of sciences, and the plentiful fountain of advantage to human affairs" (Idem, p. 348). More recently, however, Uta Merzbach and Carl Boyer in their book about the history of mathematics described the activity of the mathematicians as "(…) the formulation of statements about abstract concepts that are subject to verification by proof" (Idem, p. 1). The differences between these three points of view are striking. While the last citation conceives mathematics as a pure abstraction, a creation of the mind, independent of any application, and a science by itself unrelated to other fields, the first two suggest mathematics is an intrinsic part of knowledge, with many practical utilities, and abstractions deeply anchored in the natural phenomena. But which view is the right one?

Comparing the histories of mathematics and science we do find a trend for mathematics to start from concrete bases, developing later on into more and more abstract forms. History also shows that the development of science and mathematics, during the 6th century BCE in Greece, seemed to have followed two separate paths. However, this separation was only apparent because almost 200 years after science was created, and the religion erected by the Pythagorean around mathematics had crumbled on its own weight, the first sophisticated mathematical model of the solar system would be built in Plato's Academy, almost out of the blue. Then later, during the 3rd century BCE, mathematics would reappear in the works of Archimedes (287-212 BCE) as the basis of his method in physics (Lloyd, 1973). This was just before science almost vanished during the Roman Empire, the Romans showing few interests in such "useless" activity.

So there is a huge gap in the historical documents that hides the tight connections between mathematics and science, and when science came back in force during the 16th century, mathematics was already playing the central role. In fact, we can confidently state that it was the fusion of mathematics and science during the Renaissance that produced the Scientific Revolution. This marked the creation of modern science and the beginning of a new way of seeing the world, on which our present society would be build. Since then, the connection between mathematics and science has never ceased to grow, which has led the great Russian mathematician Nikolai Ivanovich Lobachevsky (1793-1856) to claim in the 19th century that, "There is no branch of mathematics, however abstract, which may not some day be applied to phenomena of the real world" (Merzbach & Boyer, 2011, p. 483). Understanding why this is so should help us clarify what is the nature of mathematics and better understand is usefulness for our society.

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Why was science created and why mathematics, at first, was not part of it

Science is a creation of the Greeks (Lloyd, 1970). It started during the 6th century BCE with the work of Thales of Miletus (624-546 BCE). Its goal was to find explanations of the natural phenomena based on reason. But why was science invented in Greece and at this particular time is a question that the historians have still difficulties in clarifying. Usually they point to the fact that the Greek economy and civilization were booming, which led to the foundations of large city-states, where most citizens were liberated from the basic needs of survival and the anxiety that comes with it. Consequently, they were free to pursue more mundane activities like, in particular, philosophizing about the nature and causes of things. However, there is an alternative explanation that is much more practical as well as being fully consistent with the economic and social reality of the Greek culture at this epoch, and this would have to do with the creation of democracy.

Although Greece at the time of Thales was flourishing, the period was not a peaceful one, as the citizens were preoccupied in finding what would be the best way to run their cities. At first they experimented with two systems, the traditional kingship and tyranny. Note that tyranny did not have the pejorative connotation that we apply today to the term. In those days it consisted for the citizens in electing men of values as their political leaders. These men had high reputations, elevated moral conducts, and were unusually intelligent. Intelligence is a quality that always has been admired by humans, and such men (or women) always appeared in different civilizations to play the same specific role independent of politics: the medicine men, the priests or shamans. In Greece they were the seven sages,[1] and Thales was one of them (Griffiths, 1996).

The sages were frequently consulted on important matters, and that includes politics. Some sages even became tyrants, usually with good results. But the problem was their successions, the sons of tyrants frequently turning despots, and noble despots turning tyrants. This is why democracy was invented. It was a remarkable organizational system, where the common people, *"demos"*, equally share the responsibility of the political power, *"kratos"*, regardless of their status (Raaflaub et al., 2008). But it was also sort of revolutionary, in conflict with ancestral traditions. In these conditions, it is not difficult to imagine that Thales, as a sage, already thought about the political situation in Greece, and the citizens would have been very much interested in knowing his opinion about democracy.

Here is the problem. The Greek tradition, like the traditions of many civilizations before them, was based on a mythology, which role is to make sense of the world, and the common belief was that things are the way they are because of the wills of the gods. Consequently, religion played a central role in Greek politics. In particular, any new law or change in the constitution of a city would have needed first to be approved by the oracle of Apollo (Miletus where Thales lived had its own oracle). But this tradition is awkward for two reasons. The first one is that the wills of the gods, whose motives were described as quasi-humans,[2] were accepted by default to be arbitraries. The second reason is that what the gods wanted was supposed to be communicated to humans through signs in nature, which only the oracles could read and interpret. This gave a lot of weight to the social status of the oracles. Therefore, it was not surprising to find that very frequently their decisions

[1] See https://en.wikipedia.org/wiki/Seven_Sages_of_Greece, and references therein.

[2] Like we find in Homer's poems *Iliad* and *Odyssey* or Hesiod's poems *Theogony* and *Work and Days,* all written around the late 8th or early 7th century BCE.

went against reason. For example, in 632 BCE slightly before Thales, the noble Cylon, based on the "revelations" of the oracle at Delphi, attempted to become tyrant of Athens. This was against the will of the demos, which causes men and women (the women playing a central role) to spontaneously revolt and, despite Cylon being supported by the forces of the tyrant of Megara, succeed in expulsing him from the city. That was one important move towards democracy (Raaflaub et al., 2008).

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What was Thales opinion then? Considering the arbitrariness of the role of religion in politics, he proposed to verify whether such tradition was rightly founded. How would he do that? He thought that it could be possible to verify if there was any evidence for the interventions of gods in nature. His method was science (or natural philosophy, as it was known at the time). It consists in searching for alternative explanations to the natural phenomena based solely on reason. If a logical explanation exists for a phenomenon, then there is no need for the intervention of gods. And if it turned out that this applies to all the natural phenomena, then the role of the oracle would be superfluous, and humans would be free to decide their own destiny. That would support democracy.

This suggests that science was created with a practical purpose, which is to make sense of the world. Since this is the usual role of religions in our society, it elucidates why science is frequently perceived as a threat to priests. Many Greek philosophers were persecuted because of their "untraditional" views about nature. On the other hand, Hypatia of Alexandria (ca. 350-415 CE), one of the rare women doing science at the time, was not lynched by a Christian mob because of religion, but because of her influence on political matters (Watts, 2006).[3] Obviously, by replacing religion, science becomes automatically important in politics. During his Egyptian campaign, Napoleon (1769-1821) was followed by an army of scientists,[4] stating that this was, "(…) for the good of the nation" (which, shortly after, would become Napoleon himself). Another example is Lenin's idea of communism, favoring science over religion as the foundation of society (Crowther, 1941). Contrary to the common belief, science is not a socially neutral activity.

This also applies to mathematics, although its role is more obscure. The idea of Thales in creating science was to replace myths with logic as a guide for actions in human affairs. This plan did not include mathematics because logic and mathematics were considered two different matters at the time. Moreover, concurrently with the development of physics, the Pythagoreans would give mathematics a mystical form (a new religion), clouding even more its connection with logic. However, such mysticism about mathematics was not something new, mathematics being at the source of all the mythologies in the first place.

[3] Hypatia was counselor to Orestes, the Roman prefect of Alexandria, who was opposing the new Cristian bishop Cyril in a fight for political power.

[4] They would discover the Rosetta stone, which is the key to decipher the Egyptian hieroglyphs.

Mathematics as the origin of all mythologies

Researches in neuroscience reveal that logic and mathematics are the natural consequences of the development of intelligence (Nieder, 2016). Since intelligence is the product of the activities of neurons, and neurons are found in all animals then it is not surprising to observe evidences of both intelligence and mathematical activities in them (Milius, 2016; Nieder, 2016). However, due to the higher complexity of its brain, there is one specific characteristic that makes humans unique: this is the only animal whose survival depends completely on its cultural behavior (Conroy & Pontzer, 2012).

In paleoanthropology, culture is defined as a system of shared meanings, symbols, customs, beliefs and practices that are used to cope with the environment. An important trait of culture is that it is learned by imitation or teaching. Humans do that by sharing information through an elaborate system of communication that involves special structures in the brain related to symbolic thinking and language. Without these two capacities our social behaviors and interactions would be very much limited, and the formation of large organized social groups, on which our survival depends, would not be possible.[5]

In humanoids, the first unambiguous evidences of symbolic behavior appeared with Homo Sapiens about 40 000 years ago. Those are sophisticated tools and weapons (bows, arrows and spear throwers), personal adornments, art (painted animals and humans on the walls of caves), and musical instruments like flutes.[6] We also find female figurines and elaborate graves, which suggest mystical beliefs and ritual practices were common. Later on, we find weaved baskets, fire ceramics and potteries, all with different geometrical shapes and decorations. But the most direct evidences for mathematics are bones with non-random grouping of notches, the first records of counting (Simonyi, 2012). Although identifying what these notches count is not straightforward, the most probable explanation is that they are consecutive days in lunar calendars.[7] And that leads us to the first discovery of mathematics about nature and the origin of mythologies.

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By observing attentively the sky, we distinguish the sun, the moon, the stars, the Milky Way and the planets. They all move uniformly, except for the planets that move slightly more erratically (planets means wanderers). By "measuring" their positions the first thing that humans noted is that they define regular cycles in time. This phenomenon was recorded by different cultures in different places in the world. Two clear examples are the Mesopotamian (4000 BCE) and the Mayan in Mexico (from 2000 BCE to 250 CE), who both developed numerical systems based on the number of days in a solar year; the latter chose 360, while the former adopted 60, which we still use to count minutes, seconds and angles. From these cycles our ancestors concluded that there is order in the universe.

[5] It is suggested that hunting and defense were the main environment pressures on humans to form large groups.

[6] See https://en.wikipedia.org/wiki/Paleolithic_flutes, and references therein.

[7] One good example is the Ishango bone. See https://en.wikipedia.org/wiki/Ishango_bone.

The Greek philosophers later would identify this as "cosmos", which means "beautiful order", the contrary of chaos.

Moreover, by comparing the cycles of the cosmos with events in their environments, humans found that they were correlated. Different plants grow, reproduce and die during specific periods within a year. Birds, fishes and herds of animals hunted by humans also migrate with regularity at precise epochs. A classic example is the flood of the Nile, which, once its cycle measured, brought great abundance to the ancient Egyptian civilization for at least 3 000 years. It is these correlations with the order of the cosmos that suggested there was a direct connection between what happens in nature and the destiny of humanity. This is where the notion of gods as the source of order dictating human affairs came from. It came from mathematics. Protagoras (490-420 BCE) would epitomize this mathematical metaphysics in a catchy formula, which is that "man is the measure of all things".

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And that brings us back to Pythagoras developing a "new" religion centered on mathematics. Historians are keen to remind us that at the same epoch as Pythagoras (570-495 BCE), three other charismatic personages also created new types of "religions", Laozi around 531 BCE, Buddha (563-480 BCE) and Confucius (551-479 BCE). However, the idea on which the Pythagorean religion was founded is unique, because it did not emanated from a meditation about the human condition and behavior (or moral), but rather from a mathematical reflection about nature itself. As such this belief is much more physical than metaphysical. What the Pythagoreans discovered was order at different scales than in the cosmos, and all expressible by numbers. They discovered natural crystals that show symmetrical geometric structures. They call them the five regular solids: the tetrahedron (4 faces), the cube (6 faces), the octahedron (8 faces), the dodecahedron (12 faces) and the icosahedron (20 faces). As we know now, the shapes of crystals reflect the arrangement of their atoms at the macroscopic level (Ihde, 1984), so it is the interactions between atoms that are the source of this order. Of course, the Pythagoreans had no way to know that, but they also experimented with strings in tension, using musical instruments, and were able to quantify this order. They determined that harmonious notes are produced when the length of a string is in proportion of small integers; for example, the octave is the ratio $1{:}2$, the fifth is $2{:}3$, the fourth is $3{:}4$, etc. (Simonyi, 2012, p. 50). It was from these observations and experiments that they concluded numbers are real objects and mathematics is the principle of all things.

They then started studying the numbers themselves, discovering various relations. Although many of these had no significance (this is more numerology than theory of numbers), some turned out to be valuable. For example, they considered the number 10 as special, because it is the sum of the first four integers: $10 = 1 + 2 + 3 + 4$. Why 10? We do have 10 fingers, which is a natural base for a numerical system. But they also find 10 in other forms in nature. By arranging the 4 first integers in a triangle they form the "holy tetractys", which is the basis for the tetrahedron. They also noted that the number 6, the number of faces of a cube, is the sum of its three divisors: $6 = 1 + 2 + 3$, and call all numbers with this property perfect numbers. They also call prime a number that has no other divisor than itself, like 2, 3, 5, 7, 11, 13, etc., and even find a connection between the prime and the perfect numbers: if the number $(2^n - 1)$, where n is an integer, is a prime number, then $2^{n-1}(2^n - 1)$ is an even perfect number. This rule is one of the theorems in Euclid's Elements, which would appear 300 years later (Merzbach & Boyer, 2011).

But one of the most profound discoveries of the Pythagoreans was the incommensurability of numbers. This discovery would have such an impact on their intellect that it would destroy their religious order. It started with the theorem from which the name Pythagoras is known, which allows to calculate the size of the hypotenuse, h, of a right triangle from the sizes of its two legs, x and y, using the relation $h^2 = x^2 + y^2$. The Egyptians and Mesopotamians knew about this theorem long before Pythagoras. However, it was the systematic study of numbers by the Pythagoreans that revealed its deeper meaning. They reasoned that if x and y are integers, so are their squares, and, consequently, the sum of integers should produce another integer (this is the closure of the addition operation). But, contrary to *x* and *y*, the root square of this sum is neither an integer nor a quotient of integers. There is a simple proof in one of Aristotle's books (384-322 BCE) that confirms this result based on logic. Therefore, if the logic is correct, and the Pythagoreans assumption that numbers are real objects applies, this would point to a new trait of reality that was not perceived before, which is the incommensurability of nature. Of course, this is not how the Pythagoreans understood the problem, which they saw as a failure of their logic. Why? It is because the incommensurability of nature violates their belief of a simple, beautiful, rational order. So, man is not the measure of all things after all!

Fortunately, this will not be the position adopted by the following mathematicians, although it will take them almost 2 000 years to make sense of what we now call "irrational numbers" (a term introduced by Euclid). By admitting irrational numbers as a reality, new sets of mathematical operations and constructions become possible, some of them leading, despite the higher level of abstraction, to a new understanding of matter and the universe. The Pythagoreans did not make much of the irrational numbers, but today physics would make no sense without them (π, the epitome of the irrational numbers, is ubiquitous in our understanding of nature).

To better grasp how revolutionary was this discovery we must examine how science and mathematics developed after Thales and the Pythagoreans. The program of Thales ran almost unabated during 200 years. History retained the works of at least ten physicists contributing to the project, showing that, despite being separated in space and time, this was really a collective effort, the followers criticizing the results of their predecessors, but also adding something new to the discussion. At the same time an almost equal number of men worked in developing mathematics,[8] half of them being physicists themselves, and here again the effort was collective, deducing new consequences from previous propositions and constructing new ones. The number of mathematicians and scientists active during the pre-Socratic period was so great that it would be

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[8] The Pythagoreans included women in their researches, but they disappeared with the sect.

surpassed only after the Renaissance (which started in 1300). The ideas proposed and discussed by the physicists were astonishingly moderns, ending up with the atomic theory, which proposes that matter is formed of indivisible particles in movement in the void (Leucipus of Miletus, ca. 435 BCE and Democritus of Abdera ca. 410 BCE). In summary, concerning the main goal of science, the conclusion was that there is no evidence of any intervention of gods in nature.

However, in reaction to that bold conclusion there were also acute discussions about the validity of the information gathered through the senses. In part, these discussions were based on the notion of mathematical infinite (Zeno of Elea 490-430 BCE), but mostly they were founded on the mathematical logic exposed by the proofs of geometric propositions that were developed after Pythagoras. This last point formed the creed of the metaphysics of Plato (428-374 BCE), who, impressed by Pythagoras, believed that mathematical logic leads to the absolute truth, who he assumed was infinite and unchanging, and which he called the "divine".

Although the reason why Plato adopted such a metaphysical belief is not clear to the historians, it probably has something to do with the Pythagoreans failure. Here is one possible explanation. The Pythagoreans thought that numbers are real things and the order, consequently, was in nature itself. This was consistent with their tradition and mythology, which is the cosmos. But then they discovered that contrary to their expectations the logic of mathematics, its rational, leads to a contradiction, irrational numbers. What Plato might have noticed, however, is that logic was not wrong, it was just pointing to a deeper view of reality beyond the appearances. So he concluded that the order was not in nature but in logic itself. Therefore, while Thales and the other physicists concluded, based on reason, that there is no evidence of the interventions of gods in nature, Plato (and Aristotle after him) was affirming the exact opposite, that the divine order is visible through logic. This is the message that will pass to the medieval scholars (thanks to the prolific work of Aristotle), and adopted by the early Christian church as a possible way to prove their faith using reason.[9]

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The roots of mathematics in reality

What is the nature of the truth of logic in mathematics? In his book "A Cultural History of Physics" Kàroly Simonyi stated the following "(…) according to Euclid (…) and later geometers, the [mathematical] axioms were true because they could be immediately understood and were self-evident, requiring no further proof" (Simonyi, 2012, p. 12). Then he added that this differ from the way the truth in physics is reached, "(…) the basic equations, or axioms, of the axiomatized subfields of physics (…) are true not because they can be immediately understood, but rather because the

[9] That program, however, would never work; explanations about nature based on the Bible leading to illogical statements. But that would also keep science alive.

inferences drawn from them agree with reality (…)" (*Idem*). However, this contradicts the fact that the mathematics used in physics is the same as the mathematics developed by the mathematicians, whatever the level of abstraction. Moreover, it does not explain why mathematics always seems to anticipate physical reality, far beyond what is perceived by our senses. Two remarkable examples (but there are many others) are the discovery of non-Euclidean geometry by Bernhard Riemann (1826-1846), which led to Einstein's General Relativity, and the highly abstract mathematical nature of quantum mechanics that led Dirac (1902-1984) to discover antimatter.

To help us understanding what are the differences of logic in mathematics and physics, what we need is to compare simple examples in each discipline. Let start with Euclid's Elements, which is the most renowned mathematical work in history and a monument of logical mathematics. It is also extremely modern, agreeing with the definition of mathematics as a purely abstract activity. Euclid himself saw it in this way. There is a story which reports that in response to a student who complained the study of geometry was useless, Euclid asked one of his slave to pay the student 3 pence each time he study, "since he needs to make gain of what he learns" (Merzbach & Boyer, 2011, p. 91).

Euclid's work is a collection of definitions, postulates, and propositions (theorems and constructions) with mathematical proofs. It is composed of 13 books, covering elementary and solid geometry, the theory of numbers, and the incommensurable, for which he used the term "irrational". In the first book we find five postulates in plane geometry, which truths, according to Euclid, are self-evident. The 1st one states that only one straight line-segment can be drawn between two points. Indeed, it is easy, drawing the line with a straightedge, to admit this is true without asking for a proof. And so it seems is the 2nd postulate, which states that the line-segment constructed in the 1st postulate can be extended into an infinite straight line. The 3rd postulate then states that if we use $1/2$ this segment, we can draw a circle using a compass with one point fixed in the middle of the segment and the other rotating $360°$. To each radius, r (or diameter $= 2r$), we associate one and only one circle. Then, if we trace two diameters of this circle perpendicular to each other, producing 4 right angles, and draw 4 segments between the points of contact of the diameters with the circle, h, we obtain 4 right angle triangles that are congruent, meaning one triangle can be obtained from another by a linear transformation that preserve the lengths (which is known as an isometry). Two such transformations are obvious in the construction of the 4 triangles, which are a rotation and a reflection. This is one way to verify the 4th postulate that all right angles are congruent.

As one can see, the "self-evidence" of the first four postulates comes from physical constructions. These are not abstract concepts, but descriptions of operations, experiences based on reality. This is how the Greeks developed their logic, by confirming the "truth" of their mathematical propositions based on real constructions, using a straightedge and a compass. However, the breakthrough is that once a postulate is accepted as true, then it can be used to construct different sets of propositions and theorems that are also true. One way to generalize this process is by keeping in abstract or algebraic forms the operations that the constructions represent. The way I described the 4th postulate, using isometry, is one example

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of such abstractions. Once the concept of congruence is verified geometrically, we can describe it operationally, the linear transformation becoming the congruence. This is an abstract construction based on reality.

One application of this abstraction process is the proof about the existence of irrational numbers. By construction, the two sizes of one right triangle we have drawn above are equals to the radius, r. Then using Pythagoras theorem, the hypotenuse $h^2 = r^2 + r^2 = 2r^2$. This is applying the rules of the mathematical operations on natural numbers. Then, assume h/r is a rational number, thus, $h/r = p/q$ where p and q are integers with no common factor. This is the operational definition of a rational number.

For our hypotenuse we thus get $(h/r)^2 = p^2/q^2 = 2 \Rightarrow p^2 = 2q^2$. From this we conclude that p must be even, because p^2 is an integer with common divisor 2 with q^2. This is the operational definition of an even number. A logical consequence is that since p/q has no common divisor and p is an even number, then q must be odd (not divisible by 2). So writing p in general term as an even number $p = 2_s$, where s is any integer, and substituting we get $4s^2 = 2q^2$. This means that $q^2 = 2s^2$ suggesting that q is even (recognizing the operational definition of an even number). Now, this is a logical contradiction, because a number cannot be even and odd at the same time. Here all the mathematical operations applied are true, so the logic is correct. The only thing that is false is the first assumption, which is that h/r is a rational number.

If not a rational number then what? So logic leads to a new type of number, which is the irrationals. In the above demonstration, note that the numbers were all used in general terms, based on their operational forms, numbers being objects on which apply the logical rules consistent with the set of mathematical operations $\{=,+,\times,\div,\sqrt{}\}$. The last operation is the square root $h = \sqrt{r^2 + r^2} = \sqrt{2}r$. By accepting this operation as true (operational) we get the $\sqrt{2} \approx 1.41421356$... , which is not rational, but irrational. So what we did is to extend the set of operations on the integers, making them a subset of a larger set of numbers that today we call real numbers (one operation missing is subtraction {-}, which introduces 0 as an element of the larger real number set).

Let see now how mathematics is used in the experimental method of physics. When the science of Archimedes was rediscovered, it was clear that experimentation and mathematics were indisputable parts of his method. It starts with the observation of a phenomenon in nature. The first intellectual (abstract) activity consists in isolating the characteristics that best describe this phenomenon, expressing these characteristics symbolically under the forms of quantifiable parameters. This is similar to the abstraction process in geometry. The next move is empirical, consisting in repeating the same action by varying the parameters. This produces tables of measurements that are used to determine how the parameters varied in relation with each other. Due to the uncertainty of the measurement process the data are real numbers, and the tables are similar to abstract geometrical constructions. In fact, we can produce graphics with these tables on which geometrical rules would also apply, from which new logical relations could be discerned. But what is more useful is to express the whole experience in operational form (algebraically), such that new results could be predicted before doing any extra experiments, extending in this way the set of our actions possible on reality.

To take a concrete example, we use one case studied by Archimedes, which is the buoyancy phenomenon. The parameters are the weight, volume and density of an object (w_o, V_o, ρ_o), compared to the volume of the water displaced and its weight, from which we determine the density of water (w_{wd}, V_{wd}, ρ_w). By varying these parameters empirically Archimedes found that when a body is completely or partially immersed the water exerts an "upward force" on the body equal to the weight of the water displaced by the body. This is known as the buoyancy. The apparatus used for the measurements was a balance, following the physical principle that equal weights are in equilibrium. However, the novelty is that the object to be equilibrated was immersed in water, reducing its weight by an amount quantified by the balance. In operational form this experience is described as $w_b = w_o - w_{wd}$, where w_b is the weight registered by the balance.

This experience introduces two new concepts. The first one is that the water displaced exert a force equals to the weight, implying that the weight is a force. But it would be Newton, almost 2 000 years later, who would show the weight is a force equal to $w_o = m_o\, g$, where the first parameter on the right is the mass (the quantity of matter) and the other is a constant of proportionality called the gravitational constant. When we use a balance the constant cancels out, such that balanced weights are really equal masses. Note that Archimedes was familiar with the concept of force, having experimented before with levers. But he did not had to express the force explicitly because like for the buoyancy the lever was also a case of equilibrium (the balance works on the principle of a lever).

The second new concept in Archimedes experiment is the density. Empirically the weight of an object is found to be proportional—in a specific way—to its volume, V. We can write the relation as $w_o = m_o\, g = \rho_o V_o\, g$, where ρ_o is the density, $\rho_o = m_o/V_o$. When we replace these relations in the formula for the buoyancy we get $w_b = w_o - w_{wd} = \rho_o V_o\, g - \rho_w V_{wd}\, g$, which is equal to 0 for an object that floats, implying that $\rho_o V_o = \rho_w V_{wd}$. And here is one clear result that cannot have been deduced from the mathematical logic, which is that the volume of water displaced must then be equal to the volume of the object when totally immersed. The legend said that when Archimedes realized this fact, he was taking a bath and he got so excited by his discovery that he jumped out of it, running nude in the streets of Syracuse shouting "eureka" (I have found it).

So Simonyi was right by claiming that the truth in physics is empirical, it comes from constraints on our interactions with reality. However, he was wrong in concluding that the truth in mathematics has a different origin, since it was the abstract mathematical-operator form that led to the discovery of the physical constraint in the first place. Once the constraints are included, mathematical logic predicts new physical results that can be verified by experimentations. For example, although a massive and dense object (a ship) cannot float, reducing its density by artificially increasing its volume, displacing a greater amount of water, allows it to float.

Contrary to what is usually believed, therefore, the truth in mathematics has the same origin as in physics, both are constructed from interactions with reality.

Contrary to what is usually believed, therefore, the truth in mathematics has the same origin as in physics, both are constructed from interactions with reality. The Greeks verified how they think using geometrical constructions, which are abstract mathematical figures that they thought represent real objects in their environment. The Pythagoreans did the same with numbers, describing relations (ratio, equality, product, difference, etc.) between objects in their environment. But once the rules of logic are empirically confirmed, we can replace these constructions and numbers by their operational equivalents. This abstraction process allows mathematical logic to apply now on general objects and constructions. This is how it is done in physics. Using mathematics an experience on reality is expressed in operational form, and mathematical logic is used to deduce the consequences of new interactions. Here is the power of mathematics, extending (like imagination) our actions beyond what is presently possible.

But like imagination not all the consequences foreseen by mathematics are realizable, the abstraction process itself being limited by physical constraints. Through experiences, some abstract relations transform into empirical ones, the "natural laws" of physics revealing new aspects of reality. This explains why mathematics always seems to be one step ahead of experimentations in physics. Archimedes was well aware of the power of mathematics in physics, claiming: "Give me a fulcrum and I will raise the world".

One important example how mathematics extends our physical reality is related to the proof of Euclid's 5th postulate. Remember that Euclid believed the truth of his postulates to be self-evident. But in the case of the 5th one this is far from obvious. The 5th postulate describes in operational form the geometric construction of parallel lines. First, draw two lines intersecting a third. Then, measure the inner angles on one side. If their sum is different than two right angles, the two lines are not parallel; extended sufficiently the two lines will eventually intersect. Many mathematicians searched for a proof of the 5th postulate within the Euclidean geometry without success. Then, in the 19th century, the great mathematician Gauss (1777-1855) concluded there was none, although he did not proved it nor understood what this result meant (Merzbach & Boyer, 2011, p. 495). It was Riemann who found the answer in 1854. What he demonstrated to the world was that the Euclidean geometry is only one special case of a more general set of possible geometries.

Consider the Pythagorean theorem. What this theorem describes physically is how we measure the interval between any two points in space. This is known as the metric and what Riemann showed is that this metric depends on the nature of the space. In a 3-dimensional, Euclidean space, the metric has the form of a sum of three "infinitesimal" distances, $ds^2 = dx^2 + dy^2 + dz^2$. In general, however, the interval takes the form of a 3-manifold:[10]

$$ds^2 = \begin{matrix} g_{11}dx^2 + g_{12}dydx + g_{13}dzdx + \\ g_{21}dxdy + g_{22}dy^2 + g_{23}dzdy + \\ g_{31}dxdz + g_{32}dydz + g_{33}dz^2 \end{matrix} = \begin{bmatrix} g_{11}dx & g_{12}dx & g_{13}dx \\ g_{21}dy & g_{22}dy & g_{23}dy \\ g_{31}dz & g_{32}dz & g_{33}dz \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}$$

The Euclidean geometry corresponds to $g_{11} = g_{22} = g_{33} = 1$ and all the others are 0.

[10] The formula on the right is in matrix form, which gives a clearer view of the operational definition: the big matrix being an operator as applied to a vector, assuming a space in 3 dimensions.

But are the other geometries physically real? In Einstein's relativity space and time are one parameter, spacetime, forming a 4-manifold. In Reimann's geometrical description the metric of spacetime implies that $g_{11} = g_{22} = g_{33} = 1$ and $g_{44} = -1$ (3 terms for space and 1 for time). The negative sign for the time corresponds to causality; causes always happen before effects. Now, one important characteristic of the n-manifolds discovered by Riemann is that in general the "spaces" are not flat, but curved (a sphere is one example). So the Euclidean geometry is flat, but Einstein's spacetime is curved (the negative sign, causality, makes it so), and what Einstein realized is that what we understand as the force of gravity is really the curvature of spacetime. Note that we do feel gravity, this is far from imperceptible, but we do not see the fourth dimension of spacetime, and thus cannot see the curvature. But we can measure it, thus it is real.

So once again, like the irrational number, mathematics points to an extension of reality far beyond the perception of our natural senses. But how this extension works really?

The epistemology of mathematics

In their book about the history of mathematics, Merzbach & Boyer refer to the 19th century as the golden years. They explain that during these 100 years mathematics increased in abstraction by introducing non-Euclidean geometries, n-dimensional spaces, non-commutative algebras, infinite processes, and non-quantitative structures. This encouraged David Hilbert (1862-1943) to present in 1900 a list of 23 problems that he believed were necessary to complete the process of reducing mathematics to an abstract system of axioms. The second problem, in particular, asks for a proof that the axioms of arithmetic form a "consistent" system, implying that a finite number of logical steps following the axioms can never lead to contradictory results. After many years of intense work, a convincing proof was found, but surprisingly it was negative.

In 1931, Kurt Gödel (1906-1978) produced two theorems of incompleteness (Gödel, 1931). The first states that no consistent system of axioms can prove the truths of arithmetic on natural numbers, because there will always be true statements within the system that cannot be proved by it. In fact, the second theorem states that the consistency of the system itself cannot be proved by the system. A straightforward interpretation of these theorems is that reducing mathematics to a system of axioms is impossible. Note that many mathematicians and philosophers refuse this conclusion, pushing the field of what they call meta-mathematics. However, if the truth in mathematics, as demonstrated above, comes as in physics from our experiences on reality, then mathematical logic is part of our cognitive system, which is open not close, and that would "explain" Gödel theorems. The key is epistemology, the theory of knowledge itself.

In 1950, the Swiss psychologist Jean Piaget (1896-1980) published three books about "genetic epistemology", which suggests that intelligence is a construction of the brain based on our interactions with reality. Summarizing Piaget's ideas, intelligence could be defined as—the integration of the action of the subject on the object (Piaget, 1950).

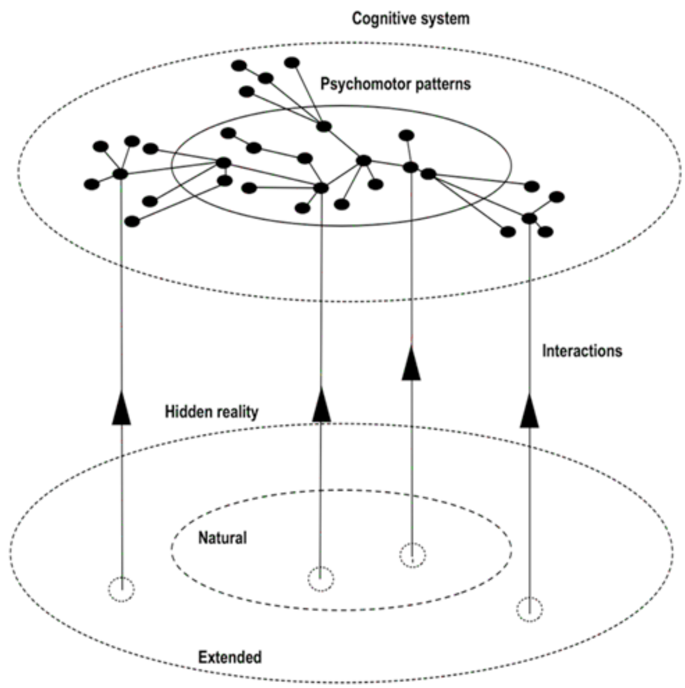

Figure 1. Illustration how the cognitive process is connected with reality in Piaget's model. Each vector represents an interaction (set of actions) with reality. These interactions are codified in the brain as patterns of actions (the dots). The relations between these patterns (the different branches) are the results of the integration process, which gives form to logic and mathematics.

The process of integration is what gives form to logic and mathematics. The model is illustrated in figure 1.[11] The object in Piaget's model is "hidden reality". The term comes from Bernard d'Espagnat's description of reality in quantum mechanics, emphasizing that "reality" is only accessible through our interactions with it (D'Espagnat, 2006). This is also the basis of Piaget's epistemological model. In the brain these interactions form psychomotor structures, or patterns of actions on reality. Concepts, ideas (or numbers) are not "things" but actions, the operational descriptions of things. The coordination of these actions (the lines linking the points in figure 1) is the integration process: abstract thinking, logic and mathematics. This process allows the brain to produce an abstract model of the world (the cognitive system), which is made of all the actions possible on reality. The goal is optimizing our actions on reality in order to increase our chance of survival.

By definition, the construction of these patterns of actions follows a long series of trials and errors. As such, therefore, the process is not deterministic, which explains why the set of possible actions predicted by this system, in our model, is larger than the set leading to effective interactions with reality (not all the points in figure 1 have a vector with origin in reality). But this is also the power of such system (the power of imagination), since it must leave a certain degree of liberty (choice of actions) for the process to be successful from the point of view of adaptability. On the other hand, what determines this adaptability is how successful our actions apply to reality itself. It is this

[11] This is an adaptation of a figure used by Lucio Russo in his book about Greek science (Russo, 2004), to explain how intelligence produces new technologies.

bootstrap connection with reality that explains why logical mathematics is not a consistent system, because it is open to reality through our experiences.

Since our senses are limited, the model at first is centered on the basic (natural) actions necessary for our survival (most animals). However, as our brain develop, the model becomes more complex, resulting in more sophisticated actions, related to our cultural behavior. This appears as different structures in the brain (Bear, Connors & Paradiso, 2016). In particular, language and symbolic thinking, necessary for mathematics, have their own structures, which are more recent than those common to all humanoid (Conroy & Pontzer, 2012). As the density of patterns of actions increases, the model becomes more abstract. As the level of abstraction increases, the coordination process produces new patterns of actions falling outside of our natural zone. Those leading to new effective interactions with reality are those that Russo identified with the source of new technological abilities. But the meaning of these structures goes deeper: it corresponds to an extension of our range of actions on reality. Knowledge is the power of our actions on reality. This is what Archimedes meant when he said that with the right fulcrum he could raise the world. Knowledge is the fulcrum.

At the same time changes happen in the mind, corresponding to the integration of the extension of reality into our abstract model. Herbert Butterfield (1957) clearly noted this phenomenon in his history of modern science. He explained that any advance in science seems to imply a transposition in the mind of the scientists. This is not automatic, because a psychological blocking (due to fear, anxiety) developed rejecting new models. This explains why new views about nature based on science are not automatically accepted. But when this barrier falls down it opens the gate to a flood of new experiences and changes.

Conclusions

Comparing with Piaget's model, therefore, we can now understand the error of Plato when he identified logical mathematics with the divine. Plato saw the two aspects of reality, as described in Piaget's model, but inverted their roles, taking the cognitive system, which is the model, as reality.

Another error would be to claim that the universe is mathematical. The relation is more complicated. Mathematics is the symbolism of our actions on reality and as such it is more a part of who we are than what the universe really is. The so-called "laws of nature" are more like abstract ways for humans to understand nature, by optimizing our actions on reality, than absolute laws that "nature" follows.

This is Plato's allegory of the cave: reality is but the shadows of things projected on the wall, and the light that reveals them is logic that emanates from the divine. But, in fact, logic is a model built from our own interactions with reality and the shadows on the wall are really our images in a mirror.

Based on Piaget's model, we can conclude that mathematics, as a consequence of the development of intelligence, is not as much a tool to extend the power of our actions on reality, as it is a way to extend our consciousness. And that is the most important role of mathematics in science and in our society.

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